EXPLICIT DIFFERENCE METHOD OF SOLVING THERMAL AND HYDRODYNAMIC BOUNDARY LAYER EQUATIONS

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A simple difference method is proposed for integrating the thermal and hydrodynamic boundary layer equations for a steady flow of compressible fluid along a permeable plate.

The system of boundary layer equations for a compressible fluid flowing along a permeable plate in the presence of heat transfer can be written in the following dimensionless form:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial \mu}{\partial y}\frac{\partial u}{\partial y} - \rho_\infty V \frac{dV}{dx} = 0, \tag{1}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{2}$$

$$\rho u \frac{\partial t}{\partial x} + \rho v \frac{\partial t}{\partial y} - \frac{1}{\Pr}\left(\mu \frac{\partial^2 t}{\partial y^2} + \frac{\partial \mu}{\partial y}\frac{\partial t}{\partial y}\right) - \theta\left[\mu\left(\frac{\partial u}{\partial y}\right)^2 - u\rho_\infty V \frac{dV}{dx}\right] = 0, \tag{3}$$

where the temperature criterion $\theta = (c_p/c_v - 1)M_\infty^2$. This system of equations can be solved exactly only in individual cases for fixed thermophysical characteristics and very simple boundary conditions. Thus, for solving most practical problems various approximate methods are employed. Numerical methods are now most frequently used for solving boundary layer problems. A number of authors [1–3] have proposed implicit difference methods for solving the equations of plane and axisymmetric boundary layers. These methods have certain disadvantages, in particular, the difficulties that arise in calculating the initial section and ensuring a smooth matching of the longitudinal velocity $u(x, y)$ and temperature $t(x, y)$ functions and the functions $V(x)$ and $T(x)$ at the outer edge of the boundary layer.

In what follows an explicit difference method is proposed for solving Eqs. (1)–(3) with the following boundary conditions:

$$u(0, y) = u_0(y), \quad v(0, y) = v_0(y), \quad t(0, y) = t_0(y), \tag{4}$$

$$u(x, 0) = 0, \quad v(x, 0) = v_c(x), \quad t(x, 0) = t_c(x), \tag{5}$$

$$\left|\frac{\partial^k [V(x) - u(x, y)]}{\partial y^k}\right|_{y\to\infty} = 0,$$

$$\left|\frac{\partial^k [T(x) - t(x, y)]}{\partial y^k}\right|_{y\to\infty} = 0,$$

$$k = 0, 1, \ldots, \tag{6}$$

where the functions $u_0(y)$, $v_0(y)$, $t_0(y)$, $v_c(x)$, $t_c(x)$, $V(x)$, $T(x)$ are assumed given.

To solve problem (1)–(6) in the plane (x, y) we introduce the rectangular network

$$x_n = nl, \quad n = 0, 1, \ldots, \quad l > 0,$$
$$y_m = mh, \quad m = 0, 1, \ldots, \quad h = \text{const}.$$

For problem (1)–(6) we substitute the following difference problem to determine the approximate values u_{nm}, v_{nm}, and t_{nm} of the functions $u(x_n, y_m)$, $v(x_n, y_m)$, and $t(x_n, y_m)$:

$$u_{n+1,m} = (1 - 2M_{nm})u_{nm} + M_{nm}(1 + N_{nm})u_{n,m-1} + M_{nm}(1 - N_{nm})u_{n,m+1} + F_{nm},$$

$$m = 1, 2, \ldots, M_{n+1} - 1, \tag{7}$$

$$v_{n+1,m} = \frac{1}{\rho_{n+1,m}}\left\{(\rho v)_{n+1,m-1} - \frac{h}{2l}\times\right.$$
$$\left.\times\left[(\rho u)_{n+1,m} - (\rho u)_{nm} + (\rho u)_{n+1,m-1} - (\rho u)_{n,m-1}\right]\right\},$$

Results of Solving the Problem of the Boundary Layer on a Flat Plate

	y^*	Results of numerical solution				Results of exact solution
		x=0.01	x=0.04	x=0.09	x=0.16	
$\frac{u}{V}$	0	0	0	0	0	0
	1	0.33557	0.33070	0.33031	0.33005	0.32979
	2	0.64071	6.63178	0.63082	0.63030	0.62977
	3	0.84692	0.84692	0.84649	0.84617	0.84605
	4	0.95080	0.95480	0.95501	0.95507	0.95552
	5	0.99116	0.99115	0.99107	0.99116	0.99155
	6	1	0.99908	0.99885	0.99886	0.99898
	7	1	1	0.99991	0.99990	0.99992
	8	1	1	1	1	1
$\tau_c = \sqrt{\frac{x}{\mu\rho V^3}}$		0.33557	0.33243	0.33236	0.33220	0.33206
M		5	13	24	37	—

$$m = 1,\ 2,\ \ldots,\ M_{n+1}, \tag{8}$$

$$t_{n+1,m} = (1 - 2M'_{nm})\,t_{nm} + M'_{nm}(1 + N'_{nm})\,t_{n,m-1} + \\ + M'_{nm}(1 - N'_{nm})\,t_{n,m+1} + R_{nm},$$

$$m = 1,\ 2,\ \ldots,\ M_{n+1}-1, \tag{9}$$

$$u_{0m} = u_0(y_m),\quad v_{0m} = v_0(y_m),\quad t_{0m} = t_0(y_m), \tag{10}$$

$$u_{n+1,0} = 0,\quad v_{n+1,0} = v_c(x_{n+1}),\quad t_{n+1,0} = t_c(x_{n+1}), \tag{11}$$

$$u_{n+1,m} = V(x_{n+1}),\quad t_{n+1,m} = T(x_{n+1})$$

$$\text{for}\ \ m > M_{n+1}, \tag{12}$$

where

$$\rho_{nm} = \rho(t_{nm}, p_n),\quad p_{n+1} = p_n - l\rho_\infty(x_n)\,V(x_n)\,\frac{dV(x_n)}{dx},$$

$$\mu_{nm} = \mu(x_n,\ y_m,\ t_{nm}),\quad M_{nm} = \frac{l\,\mu_{nm}}{h^2(\rho u)_{nm}},$$

$$N_{nm} = \frac{h}{2\mu_{nm}}\left(\frac{\mu_{n,m+1} - \mu_{n,m-1}}{2h} - \rho_{nm}v_{nm}\right),$$

$$F_{nm} = \frac{l\rho_\infty(x_n)V(x_n)}{(\rho u)_{nm}}\,\frac{dV(x_n)}{dx},\quad M'_{nm} = \frac{M_{nm}}{\Pr},$$

$$N_{nm} = \frac{h}{2\mu_{nm}}\left(\frac{\mu_{n,m+1} - \mu_{n,m-1}}{2h} - \rho_{nm}v_{nm}\Pr\right),$$

$$R_{nm} = \frac{l\theta}{(\rho u)_{nm}}\left[\mu_{nm}\left(\frac{u_{n,m+1} - u_{n,m-1}}{2h}\right)^2 - \right. \\ \left. - u_{nm}\,\rho_\infty(x_n)\,V(x_n)\,\frac{dV(x_n)}{dx}\right],$$

$$M_{n+1} = \max\{m+1,\ M_0\}\ \ \text{for}\ \ k(\Psi_{n+1} - \psi_{n+1,m}) \leqslant \\ \leqslant (\psi_{n+1,m} - \psi_{n+1,m-1}),$$

$$\Psi = V,\ T,\quad \psi = u,\ t. \tag{13}$$

Condition (13) ensures that the functions u and t and their derivatives up to the k-th order are smoothly joined with the functions V and T. It was obtained as follows. In accordance with condition (6) the absolute values of the difference derivatives of the functions u and t near the outer edge of the boundary layer must decrease monotonically, i.e., on the layer n + 1 there must exist three successive points such that

$$\left|\left(\frac{\partial^k\psi}{\partial y^k}\right)_{\gamma-2}^{n+1}\right| > \left|\left(\frac{\partial^k\psi}{\partial y^k}\right)_{\gamma-1}^{n+1}\right| > \left|\left(\frac{\partial^k\psi}{\partial y^k}\right)_{\gamma}^{n+1}\right| = 0,$$

$$k = 1,\ 2,\ \ldots\ . \tag{14}$$

The approximate value of the k-th order derivative is found with the following difference relation:

$$\left(\frac{\partial^k\psi}{\partial y^k}\right)_m^{n+1} = \\ = \frac{1}{h}\left[\left(\frac{\partial^{k-1}\psi}{\partial y^{k-1}}\right)_{j+1}^{n+1} - \left(\frac{\partial^{k-1}\psi}{\partial y^{k-1}}\right)_j^{n+1}\right], \tag{15}$$

where

$$j = \begin{cases} m & \text{when } k = 2\nu - 1 \\ m - 1 & \text{when } k = 2\nu. \end{cases}\qquad \nu = 1,\ 2,\ \ldots .$$

From conditions (14) and (15) it follows that

$$\gamma = \begin{cases} M_{n+1} + \dfrac{k-1}{2} & \text{when } k = 2\nu - 1, \\ M_{n+1} + \dfrac{k}{2} & \text{when } k = 2\nu, \end{cases}$$

$$\left(\frac{\partial^k\psi}{\partial y^k}\right)_{\gamma-1}^{n+1} = \frac{(-1)^{k-1}}{h^k}\,(\Psi_{n+1} - \psi_{n+1,M-1}),$$

$$\left(\frac{\partial^k\psi}{\partial y^k}\right)_{\gamma-2}^{n+1} = \frac{(k-1)(-1)^k}{h^k}(\Psi_{n+1} - \psi_{n+1,M-1}) + \\ + \frac{(-1)^{k-1}}{h^k}\,(u_{n+1,M-1} - u_{n+1,M-2}).$$

Condition (13) follows directly from the expressions for the difference derivatives $(\partial^k\psi/\partial y^k)_{\gamma-1}^{n+1}$ and $(\partial^k\psi/\partial y^k)_{\gamma-2}^{n+1}$.

It should be noted that the use of the proposed method of matching functions is not confined to the boundary layer problem. Relation (13) can also be used, for example, in the numerical solution of problems of the temperature field in an infinite or semi-infinite mass.

The procedure for solving difference problem (7)–(13) is as follows. The values u_{0m}, v_{0m}, t_{0m}, m = 1, 2, ... are determined from conditions (10). If the calculations are made starting from the leading edge of a plate exposed to a flow with equalized velocity and temperature fields, then, $u_{0m} = V(0)$, $v_{0m} = 0$, $t_{0m} = T(0)$, m = 1, 2, ... By proceeding stepwise along the x-axis we can determine the network functions u, v, and t for the entire region in question. In fact, we will assume that u_{im}, v_{im}, t_{im}, i = 1, 2, ..., n have already been found, and it is required to determine them for i = n + 1. First, we find the values $u_{n+1,m}$ and $t_{n+1,m}$ from conditions (7), (9), and (11), checking that the inequality of (13) is satisfied for all $m > M_0 \geq 0$. If this inequality is satisfied, further computation of the velocities and temperatures from (7) and (9) on the layer n + 1 ceases, since condition (12) enters into force. The transverse velocity on the layer n + 1 is calculated from conditions (8) and (13) successively for m = 1, 2, ... This velocity becomes constant at $m \geq M_{n+1}$. The number M_0, which affects the error in computing the network functions near the leading edge of the plate, should be selected in the range from 1 to 3.

An investigation of the convergence of the solution of problem (7)–(13) to the solution of the system of nonlinear differential equations (1)–(6) presents considerable difficulties. However, the necessary convergence conditions can be obtained as follows. For convergence it is necessary that the difference scheme consist of equations, the solution of each of which converges to the solution of the corresponding differential equation and is consistent if all the unknown functions but one have been exactly determined.

We assume that the velocity field is given in the region in question. Then for convergence of the solution of the difference equation of heat propagation (9) to solution (3) the following conditions must be satisfied [4]:

$$0 \leqslant M'_{nm} \leqslant 0.5,\quad |N'_{nm}| \leqslant 1. \tag{16}$$

We further assume that the transverse component of the velocity vector and the thermophysical characteristics are given functions of the coordinates, while the solution of Eq. (1) with certain given boundary conditions is a sufficiently smooth function satisfying the condition $u(x, y) \geq C_1 > 0$. By C_i, $i = 1, 2, \ldots$ we denote positive constants. Then we can show that if the conditions

$$0 \leqslant M_{nm} \leqslant 0.5, \ |N_{nm}| \leqslant 1 \tag{17}$$

are satisfied, the network function u_{nm} converges to the exact solution as $h \to 0$. In fact, in accordance with our assumptions, Eq. (1) can be written in the following form:

$$\begin{aligned} u(x_{n+1}, y_m) &= [1 - 2M(x_n, y_m)]\, u(x_n, y_m) + \\ &+ M(x_n, y_m)(1 + N_{nm})\, u(x_n, y_{m-1}) + \\ &+ M(x_n, y_m)(1 - N_{nm})\, u(x_n, y_{m+1}) + \\ &+ F_{nm} \frac{u_{nm}}{u(x_n, y_m)} + lK, \end{aligned} \tag{18}$$

where

$$M(x_n, y_m) = M_{nm} \frac{u_{nm}}{u(x_n, y_m)}, \ K = 0(l + h^2) \leqslant lC_2.$$

Since

$$\frac{\varepsilon_{nm}}{u_{nm}} = \frac{\varepsilon_{nm}}{u(x_n, y_m)} \left[1 + \frac{\varepsilon_{nm}}{u(x_n, y_m)} - \left(\frac{\varepsilon_{nm}}{u(x_n, y_m)}\right)^2 + \ldots \right] \leqslant \frac{\varepsilon_{nm}(1 + C_3 \varepsilon_{nm})}{u(x_n, y_m)},$$

where $\varepsilon_{nm} = u(x_n, y_m) - u_{nm}$, using conditions (7) and (16) we obtain

$$\begin{aligned} |\varepsilon_{n+1,m}| &= |(1 - 2M_{nm})\varepsilon_{nm} + M_{nm}(1 + N_{nm})\varepsilon_{n,m-1} + \\ &+ M_{nm}(1 - N_{nm})\varepsilon_{n,m+1} + \\ &+ M_{nm} \frac{\varepsilon_{nm}}{u(x_n, y_m)} \{2u(x_n, y_m) - \\ &- u(x_n, y_{m-1}) - u(x_n, y_{m+1}) + N_{nm}[u(x_n, y_{m+1}) - \\ &- u(x_n, y_{m-1})]\} - F_{nm} \frac{\varepsilon_{nm}}{u(x_n, y_m)} + lK \Big| \leqslant \\ &\leqslant \delta_n \Big[1 + \frac{l\mu_{nm}(1 + C_3\delta_n)}{\rho_{nm} u^2(x_n, y_m)} \left(C_4 + \frac{N_{nm}}{h} C_5\right) + \\ &+ \frac{l\rho_\infty(x_n) V(x_n)(1 + C_3\delta_n)}{\rho_{nm} u^2(x_n, y_m)} \frac{dV(x_n)}{dx}\Big] + \\ &+ l^2 C_2 \leqslant \delta_n [1 + l(C_6 + C_7\delta_n)]. \end{aligned}$$

Here, δ_n is a number not less than the maximum of the absolute value of the error ε_{nm} for layer n; $C_4 = \max|\partial^2 u/\partial y^2|$; $C_5 = \max|\partial u/\partial y|$. From the latter inequality there follows the convergence of the network function to the function $u(x_n, y_m)$ with the given assumptions. The steps l and h of the difference network are selected on the basis of conditions (14) and (15).

Difference scheme (7)–(12) was numerically tested by solving a number of hydrodynamic and thermal boundary layer problems on a BESM-2M computer. In the calculations the dynamic viscosity μ was assumed to be a function of temperature, and the density ρ inversely proportional to the temperature.

Numerous calculations, in which the quantities Pr, μ, ρ, v_c, t_c, θ were varied, showed that violation of conditions (14) and (15) makes the computation process unstable.

The accuracy of solutions obtained by the proposed difference method can be judged from a comparison of the numerical solution and the known exact solution for the problem of the boundary layer on a plate in a stationary homogeneous external flow [5]. The table presents the results of solving this problem for the following starting data: $u_0 = V = 1$; $v_0 = v_c = 0$; $t_0 = T = t_c = 1$; $\theta = 0$; Pr = 1; $\rho = 1$; h = 0.01; $l = 0.0001$. It is clear from the table that the errors in computing the velocities, which are presented as a function of the dimensionless coordinate $y^* = y\sqrt{V\rho/\mu x}$, decrease in the direction of the x-axis. The same applies to the error in computing the friction stress on the plate surface τ_c, which is calculated from the formula

$$\tau_{cn} = \mu \frac{u_{n1} - u_{n0}}{h}.$$

The error in computing τ_c at x = 0.16 does not exceed 0.05%. It should be noted that the rate of increase of the number of steps M along the y-axis decreases rapidly along the flow.

NOTATION

u and v are the longitudinal and transverse components of the velocity vector; ρ is the density; μ is the coefficient of dynamic viscosity; V, T, and ρ_∞ are the longitudinal velocity, temperature, and density at the outer edge of the boundary layer; Pr is the Prandtl number; θ is the temperature criterion; c_p and c_v denote the specific heats at constant pressure and volume; M_∞ is the Mach number; τ is the friction stress; y* is the dimensionless coordinate. Subscripts: 0) parameters in the initial section of the boundary layer; c) parameters at the surface of the plate.

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